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NON-SMOOTH OPTIMIZATION METHODS IN THE GEOMETRIC INVERSE GRAVIMETRY PROBLEM

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Abstract. The 2D localization inverse problem of mathematical geophysics is considered. It is of kind of so-called gravimetric problems, due to the underlying process of primary data gathering, the gravimetry. The latter is systematically fulfilled measuring of the gravitational field on the Earth's surface with special high-precision digital appliances, gravimeters. By virtue of the known Poisson equation, which ties together the gravitational potential and material density, the measured fluctuations of potential gravitational field might point on the increase or decrease of material density under Earth's surface, therefore, indicating, for example, desired mineral fields, or underground voids, filled with liquid or gas (which is not always desired, and even can be dangerous). Here we have to accomplish a coordinate recovery for homogeneous gravitational anomaly by the results of gravimetry. In this work we show that minimization process here isn't trivial: the functional has a derivative in any direction, but its classic gradient does not exist. So, then we should switch on non-gradient methods instead. For the numerical solution of the problem, we applied subsequently the subgradient, Nelder-Mead, and the genetic algorithms. Based on results of these computations, a comparative analysis of the applied algorithms is conducted. Also, here we propose corresponding subgradient esteems and the ready-to-use subgradient formulae for numerical solving of this problem.

Keywords: Inverse problems, ill-posed problems, conditions on part of boundary, Poisson equation-based model, non-smooth functional, non-gradient algorithms, oil industry.

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1 Introduction

It is generally known that long-lasting oil and gas field mining can lead to the negative consequences. The main geodynamic negative effects include deformation and seismic. Deformation effects of mining are subdivided on vast drawdowns of earth surface and anomaly activities of the fault zones that are located within the field. Seismic effects appear as technogenic earthquakes. It is almost impossible to make reliable *a priori* conclusion to determine the near future scenario and form of local geodynamic events, hence this problem requires an analytic approach. Potential forms of natural or half-technogenic emergencies induce the choice of adequate basic monitoring system, which will help to predict and plan the response on such events throughout the whole period of hydrocarbon field development. One of such a method is gravimetric monitoring, which has sense in esteem of the deformation processes into the productive deposits (Virtanen, 2006; Sorokin, 1953; Grushinsky, 1961). Gravimetric monitoring is a system of gravity observation along with primary raw data pre-processing. Its core feature is gravitational acceleration metering over a region of Earth surface with gravimeters, special digital instruments of high precision (Vinogradov & Bolotnova, 2010). Mathematical models of considered geophysical systems provide necessary theoretical basis for gravimetry data processing. As we know, gravitational field is of kind of potential fields, and this field is characterized by the scalar function of gravitational field potential (Misner et al., 1973; Ivanenko & Sardanishvili, 2012). Within observed region, the gravitational field potential is described by Poisson equation. This equation contains density distribution function in its right-hand side. In this context, here we understand a straight gravimetry problem as computation problem of finding gravitational field potential function (and its derivatives) by known density distribution within mineral field. Correspondently, we consider an inverse gravimetry problem as a problem of mineral field density distribution recovery by gravimetric data, a package of current meterage information about appropriate gravitational field (Grushinsky, 1983). Such an information is provided by high-precision gravimeter measurements of gravitational force acceleration. This acceleration equals to the gravitational field gradient with accuracy up to the sign. There exists a significant amount of works dedicated to the various aspects of inverse gravimetric problems (see, for example, (Barzaghi & Sans, 1998; Michel, 2005; Jacoby & Smilde, 2009; Prutkin & Saleh, 2009; Balk & Yeske, 2013; Pilkington, 2012; Vasin et al., 2013; Akimova et al., 2014; Balk et al., 2017; Skurydina, 2017; Vatankhah et al., 2018; Serovajsky et al., 2019a,b)). Here we consider the coordinate recovery problem for gravity anomaly by known on-surface gravitational field measuring. A substantial feature of this problem, as we'll see below, is non-differentiability of the corresponding functional, whereupon gradient methods become helpless here, and only various methods of non-smooth optimization might be applicable (see, for example, Shor, (1985); Vasiliev, (2002); Avriel, (2003); Nelder & Mead, (1965); Gladkov et al., (2006); Banzhaf et al., (1998)).

2 Problem statement

Let geological heterogeneity be located somewhere in-depth under Earth surface, and its location is unknown. We consider a problem of heterogeneity location recovery by gravimetry data gathered on-surface of Earth.

There is a rectangular area Ω with horizontal coordinate x and vertical one y, characterizing by equality

$$\Omega = \{(x, y) \colon 0 < x < L, 0 < y < H\},$$

where the area length L and the depth H are given. The border of area consists of earth surface, represented by lower side $y = 0$, and inner bound S, which corresponds to the lateral sides $x = 0$ and $x = L$ and upper side $y = H$. Let us call Ω a *search area*. In the area Ω the Poisson equation is considered

$$\Delta\varphi(x, y) = -4\pi G\rho(x, y), (x, y) \in \Omega,$$

where $\varphi = \varphi(x, y)$ is gravitational potential, $\rho = \rho(x, y)$ is density distribution in area given, and G is Newton gravitational constant. It is known that somewhere in the area considered, somewhat heterogeneity exists. Heterogeneity and environment densities ρ_1 and ρ_2 are assumed to be known. So, we have formula:

$$\rho(x, y) = \begin{cases} \rho_1, & (x, y) \in \Omega_0, \\ \rho_2, & (x, y) \notin \Omega_0, \end{cases}$$

where the area Ω_0 corresponds to the given heterogeneity. Let us call Ω_0 an *area*. Denoting by u the difference between potentials oft perturbed gravitational field in cases of presence and absence of gravity anomaly correspondently, with regards to the system linearity, we arrive at following equation:

$$\Delta u(x, y) = -4\pi G\psi(x, y), \qquad (x, y) \in \Omega, \tag{1}$$

where

$$\psi(x,y) = \begin{cases} \eta, (x,y) \in \Omega_0, \\ 0, (x,y) \notin \Omega_0, \end{cases} \tag{2}$$

and η is the difference between surroundings and anomaly densities (known value). In further, to be short, we will call the values u and η just the "potential" and "density", although really they are differences of the corresponding quantities.

The area Ω is chosen as large, as influence of gravity on its inner bound is practically absent, therefore, the potentials of perturbed and unperturbed fields thereon are equal. As a result we obtain a homogeneous boundary condition:

$$u(x,y) = 0, \quad (x,y) \in S. \tag{3}$$

On the earth surface gravitational measuring is realized. Thus, the values of function u and its vertical derivative are assumed to be known. Thereupon, we arrive at boundary conditions:

$$u(x,0) = \alpha(x), \ 0 < x < L, \tag{4}$$

$$\frac{\partial u(x,0)}{\partial y} = \beta(x), \ \ 0 < x < L, \tag{5}$$

where the functions α and β (respectively, gravitational field potential and its vertical derivative on-surface) considered as known. We will treat condition (4) as boundary condition in completion to (3), because the area S, being mentioned in (3), does not include the outer boundary. Also, we take into account the equality (5) as additional information which is applied to the inverse problem statement. It is assumed that the area Ω_0 is rectangle of the given size, and is characterized by equality

$$\Omega_0 = \{(x,y) \colon a \le x \le a+m, \ b \le y \le b+n\}, \tag{6}$$

where parameters m and n are known, and the center coordinates a and b are to be found. It is required to restore the location of anomaly by results of on-surface gravimetric measuring. So, considered geometric inverse gravimetry problem consists of finding such values of parameters a and b, that equations (1)-(6) were satisfied.

In accordance with general principles of solving an inverse problems (Kabanikhin, 2012) we pass to the functional minimization problem

$$I(a,b) = \int_0^L \left[\frac{\partial u(x,0)}{\partial y} - \beta(x)\right]^2 dx, \tag{7}$$

where u is the solution of the boundary problem (1), (3), (4), which depends on the parameters a and b in accordance with equalities (2) and (6).

3 Analysis of functional

Here we have two numeric parameters unknown. In this regard, the problem conceivably could be solved with simple brute force search. However, at first, in real situation, one happens to deal with enormously big areas, which leads to huge amount of computations when solving just by brute force. Second, when making clarification of our inverse problem statement in the future, this approach might be appear principally unsuitable, whereas we appreciate to have a capable algorithm for solution of problems much more complicated than this one. Third, as we will see below, this one problem possesses the mathematical features that are far enough from trivial, whereupon its qualitative analysis is the matter of serious interest.

For practical solving of optimal control problems of unconditioned extremum, gradient methods are wide and most-often applied (see, for example, Vasiliev, (2002); Banzhaf et al., (1998); Kabanikhin, (2012); Céa, (1971)). In such cases one uses derivatives of the minimized functional. Because of sought parameters are numbers, here we discuss the minimization of two variables function. The following statement is true:

Theorem 1. *At the arbitrary point (a, b), the function I has a derivative in any direction (g, h), which is given by following equality:*

$$I'(a,b;g,h) = \omega \int_a^{a+m} [p(x,b) - p(x,b+n)]\, dx|h| + \omega \int_b^{b+n} [p(a,y) - p(a+m,y)]\, dy|g|, \quad (8)$$

where $\omega = -4\pi G\eta$, and $p = p(x, y)$ is a solution of Laplace equation

$$\Delta p(x,y) = 0, \quad (x,y) \in \Omega \quad (9)$$

with boundary conditions

$$p(x,0) = 2\left(\beta(x) - \frac{\partial u(x,0)}{\partial y}\right), \ 0 < x < L, \quad (10)$$

$$p(x,y) = 0, \quad (x,y) \in S. \quad (11)$$

Proof. Let u be a solution of (1)-(4), (6), corresponding to the parameters (a, b); and let v be a solution of the same problem, which corresponds to the pair $(a + \sigma g, b + \sigma h)$, $\sigma > 0$; and g, h are arbitrary numbers. Then the following is true:

$$\delta I = I(a+\sigma g, b+\sigma h) - I(a,b) = 2\int_0^L [u_y(x,0) - \beta(x)]\, \varphi_y(x,0) dx + \int_0^L [\varphi_y(x,0)]^2\, dx, \quad (12)$$

where $\varphi = v - u$, and corresponding partial derivatives are denoted by u_y, and so on. Obviously, function φ satisfies Poisson equation

$$\Delta\varphi(x,y) = -\omega(\chi_\sigma - \chi_0), \ (x,y) \in \Omega \quad (13)$$

with an homogeneous boundary conditions, where χ_0 and χ_σ are characteristic functions of sets Ω_0, and

$$\Omega_\sigma = \{(x,y)\colon a+\sigma g \le x \le a+\sigma g+m, \ \ b+\sigma h \le y \le b+\sigma h+n\}.$$

Multiplying equality (13) by the arbitrary function λ and integrating over area Ω, after integrating by parts with using boundary conditions, we obtain

$$\int_\Omega \varphi\Delta\lambda\, d\Omega + \int_0^H \left(\varphi_x\lambda\Big|_{x=L} - \varphi_x\lambda\Big|_{x=0}\right) dy + \int_0^L \left(\varphi_y\lambda\Big|_{y=H} - \varphi_y\lambda\Big|_{y=0}\right) dx = \omega\int_\Omega (\chi_0 - \chi_\sigma)\,\lambda\, d\Omega.$$

Choosing here the solution p of boundary problem (9)-(11) as λ, we have

$$2\int_0^L [u_y(x,0) - \beta(x)]\, \varphi_y(x,0)\, dx = \omega\int_\Omega (\chi_0 - \chi_\sigma)\, p\, d\Omega.$$

As a result, equality (12) takes form:

$$\delta I = \omega \int_{\Omega} (\chi_0 - \chi_\sigma)\, p\, d\Omega + \int_0^L [\varphi_y(x,0)]^2 \, dx. \tag{14}$$

Let us consider first integral in right-hand side of (14). We find

$$J_\sigma = \int_\Omega p\,(\chi_0 - \chi_\sigma)\; d\Omega = \int_{\Omega_0 \backslash \Omega_\sigma} p\, d\Omega - \int_{\Omega_\sigma \backslash \Omega_0} p\, d\Omega.$$

The shapes of integrating areas in the right-hand side of this equality depend on signs of g and h. Let us assume that these quantities are positive. Then we get

$$J_\sigma = \int_{a+\sigma g}^{a+m+\sigma g} \int_{b+n}^{b+n+\sigma g} p\, dxdy + \int_{a+m}^{a+m+\sigma g} \int_{b+\sigma h}^{b+n} p\, dxdy - \int_a^{a+m} \int_b^{b+\sigma h} p\, dxdy - \int_a^{a+\sigma g} \int_{b+\sigma h}^{b+n} p\, dxdy.$$

Dividing obtained expression by σ and passing to the limit, as $\sigma \to 0$ with respect to mean value theorem, we have

$$\lim_{\sigma\to 0} \frac{J_\sigma}{\sigma} = h \int_a^{a+m} [p(x,b) - p(x,b+n)]\; dx + g \int_b^{b+n} [p(a,y) - p(a+m,y)]\; dy.$$

The same way, for a negative value g, and positive h, we get

$$\lim_{\sigma\to 0} \frac{J_\sigma}{\sigma} = h \int_a^{a+m} [p(x,b) - p(x,b+n)]\; dx - \; g \int_b^{b+n} [p(a,y) - p(a+m,y)]\; dy.$$

For a positive g and negative h, we obtain

$$\lim_{\sigma\to 0} \frac{J_\sigma}{\sigma} = -h \int_a^{a+m} [p(x,b) - p(x,b+n)]\; dx + \; g \int_b^{b+n} [p(a,y) - p(a+m,y)]\; dy.$$

Finally, for both negative g and h we get

$$\lim_{\sigma\to 0} \frac{J_\sigma}{\sigma} = -h \int_a^{a+m} [p(x,b) - p(x,b+n)]\; dx - \; g \int_b^{b+n} [p(a,y) - p(a+m,y)]\; dy.$$

Thus, the following formula holds

$$\lim_{\sigma\to 0} \frac{J_\sigma}{\sigma} = |h| \int_a^{a+m} [p(x,b) - p(x,b+n)]\; dx + \; |g| \int_b^{b+n} [p(a,y) - p(a+m,y)]\; dy. \tag{15}$$

Let us give esteem to the second integral of equality (14). Formally multiplying equation (13) by function φ and integrating over area Ω, after integrating with respect to the boundary conditions we obtain the estimation

$$\|\varphi\| \le |\omega| \cdot \|\chi_0 - \chi_\sigma\|_*, \tag{16}$$

in the left-hand side of which there is the $H_0^1(\Omega)$ Sobolev space norm, and the norm of corresponding adjoint space is given by formula:

$$\|\psi\|_* = \sup_{\|\lambda\|=1} \left| \int_\Omega \lambda\psi\, d\Omega \right|.$$

Thus, the following inequality holds:

$$\|\varphi\|_* \le |\omega| \sup_{\|\lambda\|=1} |K_\sigma(\lambda)|,$$

where

$$K_\sigma(\lambda) = \int_\Omega \lambda\left(\chi_0 - \chi_\sigma\right) d\Omega.$$

Using the method described above, we find

$$\lim_{\sigma\to 0}\frac{K_\sigma(\lambda)}{\sigma} \;=\; |h|\int_a^{a+m}\left[\lambda(x,b)-\lambda(x,b+n)\right]dx \;+\; |g|\int_b^{b+n}\left[\lambda(a,y)-\lambda(a+m,y)\right]dy.$$

Then from inequality (16) follows

$$\lim_{\sigma\to 0}\frac{\|\varphi\|}{\sigma} \le |\omega| \sup_{\|\lambda\|=1} M(\lambda),$$

where the right-hand side of previous equation is denoted by $M(\lambda)$. Thereby, the function φ has an order of smallness $O(\sigma)$, as $\sigma \to 0$. This implies that $\frac{\|\varphi\|^2}{\sigma} \to 0$ as $\sigma \to 0$. Using the trace theorem, we conclude

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\int_0^L \left[\varphi_y(x,0)\right]^2 dx = 0. \tag{17}$$

Dividing the equation (14) by σ, and passing to the limit as $\sigma \to 0$ with respect to conditions (15) and (17), we arrive at formula (8). □

From the Theorem 1 immediately follows

Corollary 1. *The function I is non-differentiable.*

This statement follows directly from the fact that directional derivative dependence on (g, h) is nonlinear, according to (8). This circumstance interferes with application of gradient methods and other algorithmic techniques assuming calculation of derivatives (see, for example, (Vasiliev, 2002; Avriel, 2003; Kabanikhin, 2012; Céa, 1971; Polak, 1971). Thus, for considered problem, one should apply methods of non-smooth optimization that do not demand any classic derivative calculations. Such ones we could commonly mention as *non-gradient* methods.

4 Methods of non-smooth optimization

A natural generalization of the gradient method is *subgradient method* (Shor, 1985; Vasiliev, 2002). It is non-gradient, and it is based on the concept of subdifferential (Shor, 1985; Vasiliev, 2002; Ekeland & Temam, 1976). Let's recall that the vector $y = (y_1, y_2, \ldots, y_n)$ is called the *subgradient* of a function $f = f(x) = f(x_1, x_2, \ldots, x_n)$ at a point $(x_1, x_2, \ldots, x_n)$ if the next inequality is completed:

$$y \cdot (z - x) \le f(z) - f(x) \quad \forall z \in R^n, \tag{18}$$

and in its left-hand side is the scalar product of corresponding vectors. Also, the *subdifferential* $\partial f(x)$ is a total set of all subgradients of $f(x)$ at a given point.

A modulus nonlinearity that enters formula (8), straightly spotlights an applicability of subgradient technique, because the modulus, being a non-smooth function, simultaneously is a *subdifferentiable function.*

Theorem 2. *If the pair (a', b') is a subgradient of function I at the point (a, b), then the next inequalities are true*

$$\left|a'\right| \leq |A(p)|, \left|b'\right| \leq |B(p)|, \tag{19}$$

where

$$A(p) = \int_b^{b+n} [p(a, y) - p(a+m, y)]\, dy,$$
$$B(p) = \int_a^{a+m} [p(x, b) - p(x, b+n)]\, dx,$$

and p is the solution of adjoint system (9) – (11).

Proof. Let the pair (a', b') be a subgradient of function I at a point (a, b). Then, corresponding to the condition (18), the following inequality holds

$$a'(c-a) + b'(d-b) \leq I(c, d) - I(a, b)$$

for any numbers c and d. We put $c = a + \sigma g$, $d = b + \sigma h$, where σ is positive number, and g, h are arbitrary. Then, in the last equality, after division σ and passing to the limit as $\sigma \to 0$, by Theorem 1, we have

$$a'g + b'h \leq A(p)|g| + B(p)|h| \tag{20}$$

for all values of g and h.

Assuming in (20) $h = 0$, we have $a'g \leq A(p)|g|$ for any g. For positive values of this term, from the last inequality follows $a' \leq A(p)$. For a negative g we have $a'g \leq -A(p)g$ whence follows $a' \geq -A(p)$. Thus, first of inequalities (19) is really proven. To justify the second inequality (19), it is enough to put $g = 0$ in formula (20), and repeat the reasonings mentioned above. Q.E.D. □

From the Theorem 2 follows applicability of the *subgradient method* (Shor, 1985; Vasiliev, 2002) for solving the stated problem. Here is a brief review of corresponding ALGORITHM:

1. For somewhat natural number k, at the k-th iteration we calculate three values by the known vector (a_k, b_k) . These values are: (a) solution u_k of corresponding boundary problem (1)-(4), (6); (b) value of $I(a_k, b_k)$, and (c) solution p_k of adjoint system (9)-(11);

2. Then, corresponding to Theorem 2, we choose values a'_k and b'_k as random numbers that are normally distributed on the segments $[-|A(p_k)|, |A(p_k)|]$ and $[-|B(p_k)|, |B(p_k)|]$, respectively.

3. We find the next step approximations of values sought by the formulae

$$a_{k+1} = a_k - \alpha_k \frac{a'_k}{\left|(a'_k, b'_k)\right|},$$
$$b_{k+1} = b_k - \alpha_k \frac{b'_k}{\left|(b'_k, b'_k)\right|},$$

   where a corresponding vector modulus appears as fraction denominator; a factor $\alpha_k = \frac{\alpha}{\sqrt{k+1}}$; and α is somewhat positive number. Since the subgradient value does not always indicate the direction opposite to the one of functional decrease, then we accept the value I_k as $I_k = \min[I(a_{k-1}, b_{k-1}), I(a_k, b_k)]$. (Then go to step 1 and repeat the cycle till the stoppage criterion is satisfied.)

Among other non-gradient algorithms we note also the renowned *Nelder–Mead method*. Its idea consists of sequential movement and deforming of a *simplex* relatively to the extreme point. In this case we are actually talking about an unconditional minimization of a two variables function $I = I(a, b)$. The simplex here is a corresponding triangle. Calculation formulas for finding the subsequent approximation of the desired parameters have standard form (see (Avriel, 2003; Nelder & Mead, 1965)).

Third non-gradient method that we use here for an approximate solving of the given problem is *genetic algorithm* that also does not require derivative calculation (Gladkov, 2006; Banzhaf, 1998). Let's briefly observe the ALGORITHM:

1. We select a parameter family $\{(a, b)\} = \{(a^k_{ij}, b^k_{ij})\}$. Counters i, j, k are integers. Let's consider them as item numbers. Any pair (a, b) is called a "gene", and the whole set $\{(a^0_{0j}, b^0_{0j})\}$ is called an "initial population". Let's assign i for generation counter, j for item number of a given pair in whole generation, k for population item counter. If $J = j_k$ is k -th population size, then $j = 0, 1, 2, ..., J$. Initially, $j \geq 2$; $i, k = 0$. Here we mention a target function I as so-called "fitness function" that displays gene's "adaptability to the environment". Among all pairs (a^0_{0j}, b^0_{0j}) we select the "fittest individuals": the "best pairs" in a sense of the function I value criterion. As a result of gene "crossing", a genes -"descendants" are obtained. Descendants of next generation $(a^0_{1j_1}, b^0_{1j_1})$ (in general, $(a^k_{i+1,j_{i+1}}, b^k_{i+1,j_{i+1}})$), may inherit one of the parameters a or b from each of the "parents" with some given probability.

2. Along with each iteration, the population size changes due to appearance of new pairs as a result of crossing, and due to loss of pairs being eliminated by criterion of fitness function I . Then, survived "descendants" undergo "mutations" as a random changes of their components by random numbers.

3. We continue this cycle of "crossbreeding"-"elimination"-"mutation", and so on, until someone "new generation" of a given population regenerates to the initial population size, for example: $\{(a^0_{0j}, b^0_{0j})\}^{J_0}_{j=1} \to \{(a^1_{0j_1}, b^1_{0j_1})\}^{J_0}_{j_1=1}$, i.e.

   $$\{(a^0_{01}, b^0_{01}), (a^0_{02}, b^0_{02}), ..., (a^0_{0J_0}, b^0_{0J_0})\} \to \{(a^1_{01}, b^1_{01}), (a^1_{02}, b^1_{02}), ..., (a^1_{0J_0}, b^1_{0J_0})\},$$

   where J_0 is initial population size. When we stop computation with somewhat stoppage criterion, we pick the "fittest individual gene" from all resulting population, and it is what we sought for: a pair (a, b) of parameters needed.

If desired degree of accuracy is not achieved, we can just continue computation till demanded accuracy is reached.

5 Analysis of numeric results

To explore an applied algorithms effectiveness, we performed numeric experiment as a series of test calculations in a single search area Ω for various predefined exact solutions (a, b). Solutions were given as a gravitational anomaly location coordinates. The results are summarized in Tables 1-4. Table 5 shows an average error values for various algorithms.

Based on the results obtained, the following conclusions can be drawn:

- In general, all three methods provide a solution to the problem with satisfactory accuracy. The least accurate is subgradient method. It is able to recover the desired parameters with accuracy from 2% to 22%. Genetic algorithm and Nelder–Mead method are relatively close in accuracy. Genetic algorithm reconstructs a horizontal coordinate much better (about twice as much), and Nelder–Mead method is slightly better for vertical coordinate (on average, on 0.15 %).

- All three methods restore the horizontal coordinate of the anomaly significantly better than vertical, with the most significant difference for genetic algorithm (almost six times), and the least significant difference for subgradient method (about 1.14 times).

- An accurate comparison of algorithms in terms of speed is difficult, since the methods used were implemented by various software tools. However, if we take into account the number of iterations performed, the Nelder–Mead turned out to be the best, and the genetic algorithm is the worst.

- Summing up, we can say that subgradient method turns out to be the least effective here, and efficiency difference between genetic algorithm and Nelder–Mead is not so significant for giving preference to any of them.

Table 1: Error estimation for the solution ($a = 0.6, b = 0.7$).

Method	a, approximate value	b, approximate value	a, relative error	b, relative error
Subgradient method	0.563	0.670	0.062	0.043
Nelder–Mead method	0.606	0.724	0.010	0.034
Genetic algorithm	0.607	0.723	0.012	0.033

Table 2: Error estimation for the solution ($a = 0.5, b = 0.5$)

Method	a, approximate value	b, approximate value	a, relative error	b, relative error
Subgradient method	0.569	0.614	0.138	0.228
Nelder–Mead method	0.508	0.531	0.016	0.062
Genetic algorithm	0.503	0.532	0.006	0.064

Table 3: Error estimation for the solution ($a = 0.3, b = 0.8$)

Method	a, approximate value	b, approximate value	a, relative error	b, relative error
Subgradient method	0.337	0.762	0.123	0.054
Nelder–Mead method	0.304	0.822	0.013	0.028
Genetic algorithm	0.301	0.826	0.003	0.013

Table 4: Error estimation for the solution ($a = 0.8, b = 0.2$)

Method	a, approximate value	b, approximate value	a, relative error	b, relative error
Subgradient method	0.816	0.209	0.020	0.065
Nelder–Mead method	0.811	0.205	0.014	0.025
Genetic algorithm	0.804	0.187	0.005	0.045

Table 5: The average value of the relative error

Method	a	b
Subgradient method	8.57 %	9.75 %
Nelder–Mead method	1.32 %	3.72 %
Genetic algorithm	0.65 %	3.87 %

6 Conclusion

We should emphasise the main reason to leave gradient methods in favor of *non-gradients* here. If the classic derivative of I couldn't be find, but despite of this it *principally existed*, even then we could use here the standard methods. But according to the results obtained here (Theorem

1) classic derivative principally *does not exist*. There exists only directional derivative instead, that isn't classic. Then, we can't apply standard gradient methods that based on the classic derivative existence. But the next result (Theorem 2) gives us a working substitution for it: a possibility to use subgradients because of subdifferentiability of a corresponding functional. So, that are core arguments for non-gradient methods usage in this problem.

Concluding, let us say briefly about a practical sense of the study. Results of this work, an analytic conclusions and formulae presented here, combined with proper software and computing power, might serve for needs of oil & gas industry practitioners. For instance, it might be useful to monitor and detect changes in oil and gas fields. Especially it might work for reservoir structure monitoring under the long-term field operation, as well as for voids detecting in reservoirs.

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